PERISHABLE DURABLE GOODS

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Abstract. We examine whether the Coase conjecture (Coase (1972), Stokey (1981), Bulow (1982), Gul, Sonnenschein, and Wilson (1986)) is robust against a slight ability of commitment of the monopolist not to sell the durable goods to consumers. We quantify the commitment ability in terms of the rate at which the durable goods perish, while keeping the time between the offers small. We demonstrate that a slight commitment capability makes a substantial difference by constructing two kinds of reservation price equilibria (Gul, Sonnenschein, and Wilson (1986)) that refute the Coase conjecture.

In the first equilibrium, the monopolist credibly delays to make an acceptable offer. Almost all consumers are served, but only after extremely long delay. Most of the gains from trading is discounted away, and the resulting outcome is extremely inefficient. In the second equilibrium, the monopolist's expected profit can be made close to the static monopoly profit if the goods perish very slowly. By focusing on reservation price equilibria we rigorously eliminate any source of reputational effect. By using the first kind of reservation price equilibrium as a credible threat against the seller, we can construct many other reputational equilibria (Ausubel and Deneckere (1989)) to obtain the Folk theorem. Various extensions and applications are discussed.
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1. Introduction

In a seminal paper, Coase (1972) demonstrated that the market power of a dynamic monopolist stems from his ability to make a commitment, which is measured in terms of the time between the offers by the monopolist. This fundamental observation, known as the *Coase conjecture*, was later formulated into an extensive form game between a monopolistic seller of durable goods and a continuum of buyers (Bulow (1982), Stokey (1981), and Gul, Sonnenschein, and Wilson (1986)). Under a general condition, the game produces a unique subgame perfect equilibrium where the initial offer of the monopolist converges to the lowest reservation value of the consumers as the time between the offers converges to 0. The crisp prediction of the model helps us to diagnose the presence of the market power. The link between market power and the commitment capability revealed by the Coase conjecture leads to a number of important policies to remedy non-competitive behavior of the monopolist.

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The durability has two different implications for establishing the Coase conjecture. First, a durable good does not depreciate and generates utility over many periods. A consumer leaves the market after the purchase, as he has no need to buy another unit. As a result, the residual demand shrinks over time, which forces the monopolist to lower the price in order to make further sales. Second, a durable good does not perish. If the monopolist makes an unacceptable offer in any round, he will face exactly the same optimization problem in the following round. Hence, in any equilibrium pricing rule, the monopolist must make an offer which is accepted by some consumers. Otherwise, he simply wastes time, which cannot happen in an optimal pricing rule.[1]

With a few remarkable exceptions such as diamond and commercial lands, almost all durable goods are subject to depreciation and decay. We use interchangeably "decay," "perish" and "burn off" to mean an irreversible reduction of supply before sales. But, we differentiate these terms from "depreciate" which means the reduction of utility to the consumer over time, after the good is delivered. We call the model in which the good does not perish the *classic problem*, while referring to the model with perishable durable goods as the *perishable problem*.

Examples of perishable durable goods include smallpox vaccine and cement. Once delivered to a consumer, the vaccine provides protection against smallpox for the rest of his life. But, like many other organic compounds, its potency is subject to expiration. Cement provides service virtually forever once it is consumed. But, while in storage, some of the cement absorbs moisture in the air to become hardened and useless.

The strategic implications of depreciation of durable goods was analyzed by Bond and Samuelson (1984). If the good depreciates slowly, then the market outcome differs slightly from the case where the good does not depreciate. In this sense, depreciation make a "continuous" change to the market outcome. This paper rigorously examines the second aspect of durability by assuming that the good perishes slowly. In order to crystallize the impact of slow decay, we assume throughout the paper that the good does not depreciate once it is delivered to a consumer.

Despite the fact that many durable goods decay, however slowly, we routinely assume that the goods are not perishable in order to make the analysis tractable. We would like to see whether this simplifying assumption is justified. At the same time, this paper reveals the role of commitment power to the market outcome from a different perspective. The key insight of the Coase conjecture is to link the ability of commitment to market power by showing that the monopolistic power vanishes if the commitment power, measured in terms of time between the offers, disappears. By commitment, we mean an action that has an irreversible consequence. Because decay is an irreversible reduction of supply, it captures a different source of commitment.

Our main question is whether the market outcome is close to the competitive outcome, as dictated by the Coase conjecture, if the monopolist has a slight commitment power quantified in terms of slow decay. We shall focus on the dynamic monopoly market in which the monopolist can make offers very quickly so that the only source of commitment power is the slow decay of the durable goods.

[1]This reasoning relies on the assumption that the consumer's acceptance rule depends only upon the residual demand instead of the entire history of the game. The property holds in any subgame perfect equilibrium if the lowest reservation value of the consumer is higher than the marginal production cost.

To simplify the model, we assume that the rate of decay is exogenously given. That is, if y unit of supply is available at the end of this round, $e^{-\Delta b}y$ is available at the beginning of the next round where $\Delta > 0$ is the time between the rounds and $b > 0$ is the rate of decay. Naturally, we are interested in the case where $\Delta > 0$ and $b > 0$ are small.

In contrast to Bond and Samuelson (1984), we find a significant discontinuity in the set of equilibrium outcomes with respect to the rate of decay around $b = 0$ (no decay). We can construct a subgame perfect equilibrium which entails significant delay and is extremely inefficient. Also, we find another equilibrium in which the monopolist can generate a profit almost as large as the static monopolistic profit. The crisp prediction of the classic problem and the asymptotic efficiency of its equilibrium outcome may fail in a fundamental way when $b > 0$ is small.

To highlight the impact of slow decay, we examine the same game as the classic durable goods monopoly problem with the linear (inverse) market demand curve, $p = 1 - q$, where the monopolist offers p_t in period t which is accepted or rejected by consumers. After the offer is rejected, the monopolist has to wait for $\Delta > 0$ before offering p_{t+1}. The game continues until the market is cleared: either all consumers are served, or all available stock is sold. All agents are risk neutral with the same discount factor $\delta = e^{-r\Delta}$ for some $r > 0$.

In order to sharpen the comparison, we consider the case where the demand curve does not hit $p = 0$ ("gap case"): $\exists q_f < 1$ such that the market demand curve is $p = 1 - q$ for $q \in [0, q_f]$. In this case, the classic problem has a unique subgame perfect equilibrium in pure strategies, where the consumer's acceptance rule can be represented as a threshold rule. We call such a subgame perfect equilibrium a reservation price equilibrium (Gul, Sonnenschein, and Wilson (1986)), for which the Coase conjecture holds: in any reservation price equilibrium, the initial offer converges to the lowest reservation value of the consumers, and all consumers are served almost immediately as $\Delta \to 0$.

Because the marginal production cost is the benchmark for diagnosing the presence of the market power, the original Coase conjecture is stated for the case where the lowest reservation value of the consumer is equal to the marginal production cost ("no gap" case). For this reason, our focus will be at the case where $q_f < 1$ is sufficiently close to 1 so that the lowest reservation value of the consumer is close to the marginal cost, which is normalized to be 0.

The perishable property of the goods, regardless of the rate of decay, has two strategic implications. First, in the perishable problem the total amount of goods sold in an equilibrium is *endogenously* determined, while in the classic problem it must be exactly q_f. In fact, if $q_f < 1$, the terminal offer must be $1 - q_f$ in order to clear the market.[2] Then, by invoking the backward induction process from the terminal period, we construct a unique subgame perfect equilibrium outcome. As the terminal price is fixed to $1 - q_f$, so is the total number of periods. However, if the good is perishable, the total amount of goods delivered to the consumer depends upon the history of sales. As a result, the final price is endogenously determined. This flexibility allows us to construct many different subgame perfect equilibria, even if the gains from trading is common knowledge ("gap case").

[2] By terminal, we mean the round in which the market is cleared, but not the terminal period imposed by the modeler.

Second, if the good does not perish, it is never optimal to spend a round without making any sales. However, if the good perishes, the continuation game following a period without any sales is *not* the same as in the previous round because the supply of the good has decreased. It is easy to see that making no sales in one period could be part of an equilibrium strategy if the good perishes sufficiently quickly and if the demand curve is inelastic. We shall show that this intuition can hold even if the good perishes very slowly.

We construct two reservation price equilibria, which roughly form the upper and the lower bounds of the set of all subgame perfect equilibrium payoffs of the monopolist. In the first equilibrium, the monopolist's expected profit is close to 0 if $q_f < 1$ is close to 1 and b is close to 0. Interestingly, almost all consumers are served but the market outcome is extremely inefficient. The monopolist credibly delays to make an acceptable offer until the available stock reaches a target level. Because $b > 0$ is small, it takes exceedingly long periods for the available stock to reach the target level, and the consumer surplus is discounted away. While the monopolist generates profit slightly higher than what he could have made in the equilibrium satisfying the Coase conjecture, his profit is also very small.

In the second reservation price equilibrium, the monopolist's expected profit is close to the static monopoly profit. The slow decay opens up a strategic opportunity for the monopolist to credibly delay to make an acceptable offer for a significant time. If the consumer knows that an acceptable offer will arrive in the distant future, he is willing to accept a high price now. By exploiting the impatience of the consumers, the monopolist can achieve almost the static monopolist's profit.[3]

We demonstrate that the set of subgame perfect equilibria of the perishable problem can be significantly larger than the equilibrium outcome of the classic problem, even if the rate of decay is very small.[4] The equilibrium outcome in the "nearby" game is much richer than in the classic problem. The equilibrium analysis does *not* provide us with a precise benchmark against which the actual market outcome can be compared. Substantial market power does not necessarily imply substantial commitment power. Thus, the classical remedy to unravel the commitment capability of the monopolist may not be as effective as the analysis of the classic problem indicates.

The rest of the paper is organized as follows. Section 2 formally describes the model and the key results from the classic durable good monopoly problem. In Section 3 we illustrate the key difference of the perishable problem from the classic problem, and informally describe the equilibria, which are constructed in the following sections. Section 4 analyzes a market with a linear demand curve. Because the construction of the equilibrium strategy of the perishable problem is considerably involved, in Section 4.1, we explore an artificial game in order to highlight the mechanism that prompts the monopolist to delay making an acceptable offer. We calculate an equilibrium of the artificial game to reveal the critical

[3]The first equilibrium can serve as a credible threat by the monopolist against a deviation. Following the idea of constructing reputational equilibria in Ausubel and Deneckere (1989), we can sustain any level of monopoly profit as a subgame perfect equilibrium. Note that the second reservation price equilibrium shows that even without reputational equilibria, the monopolist can achieve almost the static monopoly profit.

[4]This statement must be qualified, because the set of equilibrium outcomes of the classic problem explodes when the "gap" disappears.

element of the subgame perfect equilibria of the perishable problem. In Section 4.2, we construct a reservation price equilibrium, which approximates the equilibrium of the artificial game in Section 4.1. We observe that the monopolist may spend many periods without making acceptable offers, while burning off the available stock to reach the target level. The equilibrium constructed in Section 4 seems to indicate that the monopolist's profit should be small if the monopolist has little ability to commit himself not to sell (small $\Delta > 0$ and $b > 0$). Section 5 shows the contrary by constructing an equilibrium where the monopolist can generate a large profit. Section 5.1 examines another artificial game, in which the monopolist can choose the time interval of making unacceptable offers to highlight the structure of the second equilibrium. We show that as $b \to 0$, the equilibrium monopoly profit in this game converges to the static monopoly profit. In Section 5.2, we construct a reservation price equilibrium, which approximate the equilibrium constructed in Section 5.1. Section 6 concludes the paper with discussions on extensions and policy implications.

2. Preliminaries

We consider a market where the demand curve is linear:

$$p = 1 - q \qquad 0 \le q \le q_f < 1 \tag{2.1}$$

where p is the delivery price. We regard each point in $[0, q_f]$ as an individual consumer. By consumer q, we mean a consumer whose reservation value is $1 - q$. We call q_f the size of the whole market.

We write a residual demand curve as $\mathsf{D}(q_0, q_f)$ after $q_0 \in [0, q_f]$ consumers are served. Following the convention of the literature, we shall treat two residual demands identical if they differ only over the null set of consumers. Let y_t be the amount of stock available at the beginning of period t. We assume that the initial stock is sufficient to meet all demand in the market: $y = y_1 > 1$.

Let q_t be the total mass of consumers who have been served by the end of period t. Thus, $q_t - q_{t-1}$ is the amount of sales in period t. Then,

$$y_{t+1} = \beta(y_t - (q_t - q_{t-1}))$$

for $\beta = e^{-\Delta b}$, $\Delta > 0$ and $b > 0$. We call $\Delta > 0$ the time interval between the offers, and $b > 0$ the instantaneous rate of decay. In the classic problem, $b = 0$, while $b > 0$ in the perishable problem.

Let h_t be the history at period t, that is, a sequence of previous offers $(p_1, \ldots, p_{t-1})$. A strategy of the monopolist is a sequence $\sigma = (\sigma_1, \ldots, \sigma_t, \ldots)$ where $\sigma_t(h_t) = p_t \in \mathbb{R}_+$ $\forall t \ge 1$. Let Σ be the set of strategies of the monopolist. Similarly, a strategy of a consumer q is a mapping from his reservation value, $1 - q$, the history of offers, h_t, and the present offer, p, to a decision to accept or reject. All agents in the model have the same discount factor $\delta = e^{-r\Delta}$ for $r > 0$. If he purchases the good at p in period t, then his surplus is $((1 - q) - p)\delta^{t-1}$.

Let $\{q_0, q_1, q_2, \ldots, q_t, \ldots\}$ be a sequence of *weakly* increasing numbers, which represent the sequence of the total mass of consumers who have been served by the end of period t.

Naturally, $q_0 = 0$. Let $\mathbf{Q}$ be the set of all such sequences. The monopolist's profit is

$$\sum_{t=1}^{\infty} \delta^{t-1}(q_t - q_{t-1})p_t$$

where $p_t = \sigma_t(h_t)$ and $h_t = (p_1, \ldots, p_{t-1})$.

We say that the market is cleared at $T_f < \infty$ if the monopolist meets all the demand for the first time, $q_{T_f} = q_f > q_{T_f - 1}$, or sells all remaining stock $q_{T_f} - q_{T_f-1} = y_{T_f} > 0$:

$$q_{T_f} = \min(q_f, y_{T_f} + q_{T_f-1}).$$

We know that in the classic problem, the market is cleared in a finite number of periods if $q_f < 1$ (Fudenberg, Levine, and Tirole (1985) and Gul, Sonnenschein, and Wilson (1986)) by meeting all the demand: $\exists T_f < \infty$ such that $q_{T_f} = q_f$.

Given the monopolist's strategy σ, consumer q's action is optimal if he accepts p_t in period t when

$$(1 - q) - p_t > \sup_{k \geq 1} \delta^k((1 - q) - \sigma_{t+k}(p_1, \ldots, p_{t-1}, p_t, \ldots, p_{t+k-1})) \tag{2.2}$$

and rejects when the inequality is reversed, where p_t is realized according to σ, $\forall t \geq 1$. By exploiting the monotonicity with respect to the reservation value, the classic problem allows us to write the optimality conditions of the consumers more compactly by focusing on the *critical type* $1 - q_t$, who is indifferent between accepting the present offer and the next offer:

$$(1 - q_t) - p_t = \delta\left((1 - q_t) - p_{t+1}\right)$$

where $p_t = \sigma_t(h_t)$ and $h_t = (p_1, \ldots, p_{t-1})$.

We say p_t is unacceptable if $q_t - q_{t-1} = 0$, $q_{t-1} < q_f$ and $y_t > 0$. Making an unacceptable offer in period t is essentially to open the market without serving anyone.

We can define a Nash equilibrium in terms of the monopolist's strategy $\sigma = (\sigma_1, \sigma_2, \ldots)$ that solves

$$\max_{\sigma} \sum_{t=1}^{\infty} \delta^{t-1}(q_t - q_{t-1})p_t \tag{2.3}$$

where $p_t = \sigma_t(h_t)$ and $h_t = (p_1, \ldots, p_{t-1})$ and, $\mathbf{p} = (p_1, p_2, \ldots)$ and $\mathbf{q} = (q_1, q_2, \ldots)$ satisfy (2.2). We say that σ is a subgame perfect equilibrium if σ induces a Nash equilibrium following every history. Although we state the definition in terms of pure strategies, the equilibrium involving mixed strategies can be defined in the same manner.

We shall focus on a class of subgame perfect equilibria where a consumer's strategy is characterized by a threshold rule which is a natural state variable of the game, namely the residual demand and the available stock.

Definition 2.1. *A subgame perfect equilibrium is a reservation price equilibrium, if there exists* $\mathsf{P} : [0, q_f]^2 \times [0, y] \to \mathbb{R}$ *such that*

$$p_t = \mathsf{P}(q_t, q_f, y_t)$$

with $y_t = \beta(y_{t-1} - (q_t - q_{t-1}))$ *where* p_t *is the equilibrium price offered in period* t*,* q_t *is the total mass of consumers served by the end of period* t *and* y_t *is the available stock at the beginning of period* t*.*

The Coase conjecture holds for the classic durable good problem.

Theorem 2.2 (Stokey (1981), Bulow (1982), and Gul, Sonnenschein, and Wilson (1986))**.** *Suppose that $b = 0$. If $q_f < 1$, then (generically) a unique subgame perfect equilibrium exists, which is a reservation price equilibrium in pure strategies. In any reservation price equilibrium, the initial offer of the monopolist converges to $1 - q_f$ and thus, his profit converges to $q_f(1 - q_f)$ as $\Delta \to 0$.*

Before describing and analyzing the perishable problem, let us describe the classic problem to review useful results. The optimization problem of the risk neutral monopolist for the classic problem is to choose a sequence $\mathbf{q} \in \mathbf{Q}$ to maximize the discounted expected profit subject to a couple of constraints:

$$\mathcal{V}^c(q_0, q_f, y) = \max_{\mathbf{q}\in\mathbf{Q}} \sum_{t=1}^{\infty} p_t(q_t - q_{t-1})\delta^{t-1} \tag{2.4}$$

$$\text{subject to} \quad (1 - q_t) - p_t = \delta((1 - q_t) - p_{t+1}) \tag{2.5}$$

$$\lim_{t\to\infty} (1 - q_t) - p_t = 0. \tag{2.6}$$

(2.5) is the constraint imposed by the rational expectations of the consumers, which renders p_t as a function of q_t and q_{t+1}. (2.6) implies that in order to clear the market, the "final" offer of the monopolist must be the lowest reservation value of the consumer.[5]

Let $T_f(q_0, q_f, y)$ be the total number of periods needed to make sales before serving every consumer in the market, or exhausting all available stock. If it takes infinitely many periods to serve all consumers, we let $T_f(q_0, q_f, y) = \infty$. In the classic problem, $T_f(q_0, q_f, y)$ is precisely the number of offers the monopolist makes in the game. Since the monopolist must serve a positive portion of consumers in every period,

$$q_t < q_{t+1} \qquad \forall t \geq 1$$

must hold, unless the market is cleared in period t. For the later analysis, it would be more convenient to interpret $T_f(q_0, q_f, y)$ as the number of periods to clear the market either by serving all consumers ($q_t = q_f$) or exhausting all remaining stock ($q_t - q_{t-1} = y_t$), after making the first acceptable offer at $t_0 \geq 1$:

$$T_f(q_0, q_f, y) = \inf\{t - t_0 + 1 : q_t = \min(q_f, y_t + q_{t-1})\}.$$

In the classic problem, $T_f(q_0, q_f, y)$ is exactly the total number of periods that the monopolist keeps the market open, because $t_0 = 1$. Let us summarize the key properties of the subgame perfect equilibrium in the classic problem, which will be a key building block for constructing a reservation price equilibrium in the perishable problem. Because these properties are already proved for the analysis of the classic problem, we state them without proofs.

Lemma 2.3. *Suppose that $b = 0$, and let $\mathbf{q}(q_0, q_f, y)$ be an optimal solution of* (2.4)*.*

(1) *Fix y. If $q_f \neq q_f'$, then $\nexists q \in [0, q_f]$ such that $\mathsf{P}(q, q_f, y) = \mathsf{P}(q, q_f', y)$.*
(2) *If $q_f < 1$, $T_f(q_0, q_f, y) < \infty$.*

[5] If $q_f = 1$ so that the lowest reservation value is 0, then the market opens indefinitely so that there is no "final" offer. Yet, the price must converge to 0 as $t \to \infty$.

(3) $\mathbf{q}(q_0, q_f, y)$ *is a continuous function of* (q_0, q_f, y) *if* $q_f < 1$.
(4) $T_f(q_0, q_f, y)$ *is a decreasing function of* q_0*, but increasing function of* q_f.

3. Illustration

The analysis of the perishable problem is considerably more difficult than the classic problem because the decision of the monopolist is affected not only by the residual demand as in the classic problem, but also by the amount of supply. Before plunging into a formal analysis, let us verbally describe the key idea.

3.1. Unacceptable Offer.

In the classic problem, making an unacceptable offer in period t implies that the monopolist is wasting one period, which cannot happen in any optimal pricing sequence. This observation is *not* carried over to the perishable problem. Suppose that at the beginning of period t, the monopolist has y_t units of the durable goods, which will reduce to y_{t+1} if no consumer purchases the good in period t, as depicted in Figure 3.1.

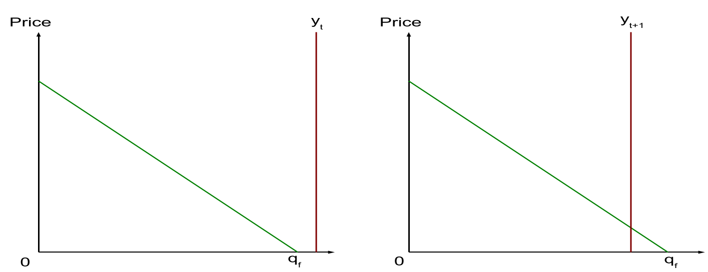

Figure 1: Left panel shows the supply of durable goods at the beginning of period t, which reduces to y_{t+1} if no consumer is served in period t. In general, however, $y_t - y_{t+1}$ is affected by how many units are sold in period t.

If y_{t+1} is close to q_f, then demand is inelastic around $q = y_{t+1}$. If the monopolist clears the market in period $t+1$, then he can generate profit of $\delta y_{t+1}(1 - y_{t+1})$, which, if δ is close to 1, is larger than $y_t(1 - y_t)$, the profit the monopolist could have generated if he cleared the market in period t.

By being able to burn off some supply, the monopolist can credibly commit not to offer any price below $1-y_{t+1}$, which is considerably higher than what he can charge if he cannot reduce the supply from y_t to y_{t+1}. If the consumers know that the future price cannot be lower than $1 - y_{t+1}$, then the monopolist should be able to sell the durable goods at a higher price now. In fact, unacceptable offers may appear along the equilibrium path, as an instrument to convince consumers of the monopolist's intention to charge higher prices in the future.

3.2. **Sell or Wait.** Even if the monopolist can generate a higher profit by intentionally burning off some supply, he needs to consider another factor before opting for this strategy. If the good perishes so slowly that it takes a long time to achieve a desirable level of supply, he would rather sell the goods now at a lower price than later at a higher price. Thus, whether or not the monopolist actually chooses to burn off some supply critically depends upon two factors: the profit from making sales now, and the time needed to achieve a desirable level of supply.

To illustrate how these two factors affect the strategic decision, let us consider two options for the monopolist, assuming that $b > 0$ is very small and the initial supply of the durable goods is $y > 1$. The first option is to follow the pricing sequence as dictated by the Coase conjecture, where the monopolist make an acceptable offer in every period until every consumer is served in $T_f(q_0, q_f, y)$ rounds. The Coase conjecture says that as the time interval $\Delta > 0$ between the offers converges to 0, $\Delta T_f(q_0, q_f, y) \to 0$. If the goods perish very slowly, the market is cleared when every consumer is served. The profit of the monopolist from this strategy would be approximately $q_f(1-q_f)$, because the price offered in period $T_f(q_0, q_f, y)$ must be $1 - q_f$, and q_f consumers are served "almost instantaneously" when $\Delta > 0$ is small.

In the second option, the monopolist keeps making unacceptable offers, burning off the supply at the rate of b until the supply reaches $1/2$ which maximizes the static monopolist profit. If $\Delta > 0$ is sufficiently small, then it will take around T^* periods where

$$e^{-\Delta b T^*} y \simeq \frac{1}{2}$$

implying

$$\Delta T^* \simeq \frac{1}{b} \left(\log y + \log 2 \right). \tag{3.7}$$

As soon as the supply reaches the target level, the monopolist clears the market at once by offering $1 - e^{-\Delta b T^*} y$ to serve $e^{-\Delta b T^*} y \simeq 0.5$ consumers. This strategy generates a discounted profit close to $0.25 e^{-\Delta r T^*}$.

Let us consider the decision problem of the monopolist at time 0. The monopolist will opt for the second option if

$$0.25 e^{-\Delta r T^*} > q_f(1 - q_f). \tag{3.8}$$

Indeed, $\forall b > 0$, we can choose $q_f < 1$ sufficiently close to 1 so that (3.8) holds. However, if $b = 0$ as in the classic problem, then $T^* = \infty$ so that this reasoning does not apply.[6]

3.3. **Time Consistency.** Suppose that (3.8) holds at time 0 so that the monopolist decides to make an unacceptable offer in period 1. An important question is whether the monopolist is willing to choose the same option by the end of period 1 as in period 0. As long as $e^{-b\Delta} y > q_f$, the answer is clear. If the monopolist opts for the Coase conjecture type strategy, his profit will be approximately $q_f(1 - q_f)$. But, if he continues to choose the second option, he can achieve the monopoly profit of 0.25 in $T^* - 1$ instead of T^*

[6]This example is only to illustrate the intertemporal decision problem of the monopolist about whether to sell now or to burn off more. We are not claiming the static monopoly profit maximizing quantity is the equilibrium target level. In fact, the equilibrium level is affected by the time preference parameter r, as it is formally described in Section 4.1.

periods. As the later analysis in Section 4 reveals, the same logic applies as long as the demand curve is sufficiently inelastic around q_f, even if $e^{-b\Delta}y \leq q_f$.

The monopolist's strategy to "burn off first and sell later" is *time consistent* in the sense that if he finds it optimal to follow the strategy now, he has incentive to do so in the next round. In fact, the time consistency is a milder form of sequential rationality, as the decision made in time 0 remains optimal for the following periods. In order to sustain the time consistent outcome path as a subgame perfect equilibrium outcome we do not need to rely on a harsh punishment off the equilibrium path as in Ausubel and Deneckere (1989). The time consistency property allows us to sustain the proposed outcome path as a stationary equilibrium outcome in which the strategy of each player is completely characterized by the residual demand and the existing supply.[7]

3.4. **Credibly Long Delay.** From (3.7), one can infer that ΔT^* increases, as $b > 0$ converges to 0:

$$\lim_{b \to 0} \lim_{\Delta \to 0} \Delta T^* = \infty.$$

While a small $b > 0$ decreases the expected profit from the strategy of "burn off first and sell later", we can reduce the profit from the Coase conjecture type strategy by selecting $q_f < 1$ sufficiently close to 1. Thus, if we choose a small $b > 0$ and $q_f < 1$ sufficiently close to 1, the monopolist can find the strategy of "burn off first and sell later" more attractive than the Coase conjecture type strategy, although the former entails a long delay. By exploiting the time consistency, we can make the long delay a credible strategy of the monopolist.

With a credibly long delay in hand, the monopolist can achieve a profit almost as large as the static monopoly profit. Suppose that the monopolist offers $p_1 = 0.5$ in the first period, which is the static monopoly profit maximizing price if all consumers with reservation value higher than 0.5 accepts the offer. Following $p_1 = 0.5$, the monopolist follows the strategy of "burn off first and sell later" which entails a credibly long delay.

Let us consider the decision problem of a consumer whose valuation is slightly above 0.5, say 0.51. He weighs the option of accepting $p_1 = 0.5$ immediately against another option of waiting for a reasonable offer which comes in the distant future (according to the strategy of "burn off first and sell later"). He will accept p_1 if

$$0.51 - 0.5 > e^{-r\Delta T^*}(0.51 - p_{1+T^*})$$

where p_{1+T^*} is the first acceptable offer in the strategy of "burn off first and sell later". By decreasing b to 0, we can make the right hand side become close to 0, which will make the consumer accept the initial offer.

We can construct a subgame perfect equilibrium in which the monopolist can accrue a profit close to the static monopoly profit, even if $\Delta > 0$ and $b > 0$ are small, i.e., even if the commitment capability of a monopolist is small. In the next two sections we formulate this intuition to construct two kinds of subgame perfect equilibria: in Section 4, one with the feature of "burn off first and sell later" and, in Section 5, another where the monopolist generates a large profit .

[7]A complete analysis is provided in Section 4.

4. Market with a Linear Demand

In order to highlight the key features of the equilibrium which will be constructed in this section, let us first examine a simple, but artificial, game. Then, we shall construct a reservation price equilibrium, which approximates the equilibrium of the artificial game.

4.1. **An Artificial Game.**

Let us consider an artificial game in which the monopolist in the market with a linear demand curve (2.1) has two options: make one final sale, or delay the sale. The monopolist can choose when he opens the market, say $\tau_f \geq 0$, and then he must make an offer to serve everyone in the market, or sell all the goods available at that point if there is an excess demand.

If the monopolist charges $p_{\tau_f} = 1 - q_{\tau_f}$ after delaying τ_f time, q_{τ_f} portion of consumers will be served. Since p_{τ_f} must clear the market,

$$1 - p_{\tau_f} = \min(q_{\tau_f}, e^{-\tau_f b} y)$$

must hold. In any equilibrium, q_{τ_f} is selected in such a way that the monopolist cannot improve his profit by delaying the sale. Define

$$h(q : \tau) = e^{-\tau r} \left[e^{-\tau b} q(1 - e^{-\tau b} q) \right] - q(1 - q)$$

$h(q : \tau)$ is the gain from delaying τ amount of real time and charging $1 - e^{-\tau b} q$ to serve everyone whose valuation is higher than $1 - q$ in the market when the present available stock is $q \leq 1$.

It is easy to see that $h(q : 0) = 0$ and

$$\frac{\partial h(q : 0)}{\partial \tau} = -(r + b)q + (r + 2b)q^2.$$

If

$$\frac{\partial h(q : 0)}{\partial \tau} \leq 0,$$

then $\forall \tau > 0$, $\frac{\partial h(q:\tau)}{\partial \tau} < 0$. If

$$\frac{\partial h(q : 0)}{\partial \tau} \geq 0,$$

then $\forall q' \geq q$, $\frac{\partial h(q':0)}{\partial \tau} \geq 0$. If $\partial h(q : 0)/\partial \tau > 0$, then the monopolist is better off if he delays sales one period. Although the total stock will be reduced to $e^{-\Delta b} q$, he can credibly charge the higher price $1 - e^{-\Delta b} q$ to generate higher profit. Similarly, if $\partial h(q : 0)/\partial \tau < 0$, then he should have accelerated sales. Thus, the optimal quantity q solves

$$\frac{\partial h(q : 0)}{\partial \tau} = 0,$$

which is

$$q = \frac{r + b}{r + 2b}, \tag{4.9}$$

and the discounted profit is

$$e^{-r\tau(y)} \frac{b(r + b)}{(r + 2b)^2}$$

where $\tau(y)$ is defined implicitly by

$$e^{-b\tau(y)}y = \frac{r+b}{r+2b}.$$

The optimal quantity (4.9) is very revealing. For a given time preference $r > 0$, if the good is not perishable ($b = 0$), then all consumers must be served, as in the classic problem. On the other hand, if the good perishes quickly (i.e., b is large), the quantity converges to $1/2$ which is the monopolistic profit maximizing quantity.

We have to verify that the delay strategy generates a higher profit than the immediate sales, from which the monopolist can generate profit $q_f(1-q_f)$ almost instantaneously if $\Delta > 0$ is small. Note that

$$e^{-r\tau(y)}\frac{b(r+b)}{(r+2b)^2} > q_f(1-q_f)$$

holds as long as q_f is sufficiently close to 1 for given b, r, y.[8] Then, a substantial delay of an acceptable offer can arise in a reservation price equilibrium.

Note that for a fixed $r > 0$,

$$\lim_{b\to 0}\frac{r+b}{r+2b} = 1$$

which implies that every consumer will be served in the limit. Yet, the outcome is extremely inefficient. A simple calculation shows that

$$\tau(y) = \frac{1}{b}\left(\log y - \log\frac{r+b}{r+2b}\right).$$

If $y > 1$, as $b \to 0$ the right hand side increases indefinitely. This implies that the monopolist is willing to delay the sale as long as possible in order to generate a positive profit, even if it is realized after a long delay. As a result, the market outcome becomes extremely inefficient because the potential gains from trading is discounted away during the long delay.

4.2. **Construction of a Reservation Price Equilibrium.** We search for a reservation price equilibrium where the equilibrium path consists of two phases. In the first phase the monopolist makes a series of unacceptable offers. In the second phase the monopolist makes a series of acceptable offers. In the second phase we can invoke the same insight as in the classic problem to construct the equilibrium path. Then, by calculating the optimal time for delay in making the first acceptable offer, we construct the equilibrium outcome where the total gains from trading vanishes as $\Delta \to 0$.

We construct an equilibrium for the rest of the section, in which the total surplus from trading is arbitrarily small, despite the fact that almost every consumer is served by the monopolist. As in Section 4.1, the monopolist delays opening the market (or equivalently, makes unacceptable offers) for $T_1 \in \{0, 1, 2, \ldots\}$ periods before making an acceptable offer. After the initial acceptable offer, the monopolist keeps making acceptable offers.

[8] One might wonder whether we have to check the same inequality for each $\tau > 0$. From the analysis of $h(q : \tau)$, we know that if this equality holds the beginning of the game, then it continues to hold for $\tau > 0$ until the available stock reaches the optimal level. Time consistency of the outcome path simplifies the analysis.

Given residual demand $\mathsf{D}(0,q_f)$ and the initial stock y with $q_f<1$, the optimization problem can be written as

$$\max_{T_1,\mathbf{q}\in\mathbf{Q}} e^{-r\Delta T_1}\sum_{t=1}^{\infty} p_t(q_t-q_{t-1})e^{-r\Delta(t-1)} \tag{4.10}$$

subject to

$$(1-q_t)-p_t = e^{-r\Delta}((1-q_t)-p_{t+1}) \tag{4.11}$$

$$p_{T_f} = 1-q_{T_f} \tag{4.12}$$

$$e^{-b\Delta T_f}\left(e^{-b\Delta T_1}y-\sum_{t=1}^{T_f} e^{b\Delta(t-1)}(q_t-q_{t-1})\right)\geq 0 \tag{4.13}$$

$$e^{-b\Delta T_f}\left(e^{-b\Delta T_1}y-\sum_{t=1}^{T_f} e^{b\Delta(t-1)}(q_t-q_{t-1})\right)(q_{T_f}-q_f)=0 \tag{4.14}$$

where T_f is the number of periods in which the monopolist make some sales. T_1 is the time during which the monopolist makes no sale, simply burning off the available stock at the rate of e^{-b}. The objective function and the first two constraints are identical to the classic problem and so is the definition of T_f.

The last two constraints warrant explanation, as they capture the key elements of the perishable problem. The first step is to observe that the trading must be completed in finite rounds, a result which is reminiscent of a well known result from the classic problem (Fudenberg, Levine, and Tirole (1985)).

Lemma 4.1. *If $q_f<1$, then in any optimal solution, $\Delta(T_1+T_f)<\infty$.*

Proof. Given the structure of the candidate equilibrium, the proof to show $T_f<\infty$ is identical to the proof in the classic problem (Fudenberg, Levine, and Tirole (1985) and Gul, Sonnenschein, and Wilson (1986)). It remains to show that $\Delta T_1<\infty$.

It suffices to show that $\exists\tau^*>0$ such that if $\Delta T_1>\tau^*$, then $(T_1,\mathbf{q})$ cannot be an optimal solution for any $\mathbf{q}\in\mathbf{Q}$.

Given the demand curve $\mathsf{D}(0,q_f)$, let $q^m(0,q_f)$ be the static monopoly profit maximizing quantity. Consider τ^* such that $e^{-b\tau^*}y=q^m(0,q_f)$. By choosing τ^*, the monopolist can charge $1-q^m(0,q_f)$, which will be accepted by all consumers whose valuations are at least $1-q^m(0,q_f)$. Thus, the equilibrium payoff of the monopolist is uniformly bounded from below by

$$e^{-b\tau^*}(1-q^m(0,q_f))q^m(0,q_f).$$

If the monopolist waits more than τ^* before making an acceptable offer, he cannot achieve this level of profit. Thus, if $\Delta T_1>\tau^*$, then T_1 cannot be an equilibrium strategy. Thus, $\Delta T_1\leq\tau^*$. □

If q_1-q_0 consumers accept the first acceptable offer from the monopolist, then at the end of the period, $e^{-b\Delta T_1}y-(q_1-q_0)$ is available, but by the beginning of period 2, only $\beta(e^{-b\Delta T_1}y-(q_1-q_0))$ is available. Thus, by the time when the market is cleared,

$$\beta\left(\cdots\left(\beta(e^{-r\Delta T_1}y-(q_1-q_0))-(q_2-q_1)\right)\right)-(q_{T_f}-q_{T_f-1})\geq 0$$

must hold, because the amount of sales in period t cannot exceed the amount of stocks available in that period. The constraint can be written as

$$e^{-b\Delta T_f}\left(e^{-b\Delta T_1}y - \sum_{t=1}^{T_f} e^{b\Delta t}(q_t - q_{t-1})\right) \geq 0. \tag{4.15}$$

Since it is possible that the market is cleared by the monopolist serving all consumers with reservation value higher than the production cost, it is possible that some goods are left over, i.e., $q_{T_f} = q_f$. However, if $q_{T_f} < q_f$, then some consumers are not served and the final offer must be such that all remaining goods are sold. Hence, the complementary slackness condition (4.14) must hold.

We show by construction that the above optimization problem has a solution.

Proposition 4.2. *Given the demand curve* $\mathsf{D}(0, q_f)$ *and initial stock* y*, there exists an optimal solution to* (4.10)*,* $(T_1, \mathbf{q})$*, which can be sustained as a reservation price equilibrium.*

Proof. See Appendix A. □

Fix y, and let $q_{T_f(\Delta)}$ be the total amount that is delivered and $T_1(\Delta)$ be the first period that the monopolist makes an acceptable offer when the time between the offers is $\Delta > 0$. Similarly, define $T_f(\Delta)$ as the round where the market clearing offer is made, since the monopolist makes the first acceptable offer at $T_1(\Delta)$. Clearly, $\forall \Delta > 0$, $q_{T_f(\Delta)} \in [0, y]$ and $\Delta T_1(\Delta) \in [0, \tau^*]$. Define

$$q' = \lim_{\Delta \to 0} q_{T_f(\Delta)}$$

and

$$\tau_1(0) = \lim_{\Delta \to 0} \Delta T_1(\Delta)$$

by taking a convergent subsequence, if necessary.

Note that the sequence of acceptable offers is precisely the same as its classic problem counterpart where the demand curve is $\mathsf{D}(0, q_{T_f(\Delta)})$. Hence, the Coase conjecture implies that the profit from the perishable problem converges to

$$e^{-\tau_1(0)b} q'(1 - q').$$

Hence, the limit properties of the reservation price equilibrium can be examined through the same method as illustrated in Section 4.1.

Let $\mathcal{W}_c(\Delta)$ and $\mathcal{W}_s(\Delta)$ be the (ex ante) expected consumer surplus and the expected producer surplus from the game where the time between the offers is $\Delta > 0$. The following proposition formalizes this observation. Because the proof is little more than the informal discussion in Section 3, we state the result without a proof.

Proposition 4.3. $\forall \epsilon > 0\ \exists \bar{b} > 0$ *such that* $\forall b \in (0, \bar{b}]\ \exists \overline{q}_f$ *such that* $\forall q_f \in (\overline{q}_f, 1)\ \exists \overline{\Delta} > 0$ *such that* $\forall \Delta \in (0, \overline{\Delta})$*,* $\mathcal{W}_c(\Delta) < \epsilon$ *and* $\mathcal{W}_s(\Delta) < \epsilon$*.*

The constructed equilibrium confirms our intuition that if the monopolist has little commitment power (small $b > 0$ and small $\Delta > 0$), then he can exercise little market power and entertain small profit. This observation is seemingly consistent with the key implication from the classic problem, which has an important policy implication. If the monopolist exercises substantial market power, then his commitment power must be substantial. Thus, by unraveling the source of the commitment power, the government can reduce the market power of the monopolist. In the perishable problem this conclusion does not hold in general because we can construct another reservation price equilibrium that generates substantial profit, despite small $b > 0$ and small $\Delta > 0$.

5. Small Commitment but Large Profit

We claim that substantial market power does not imply substantial commitment power. To substantiate this claim, we need to construct a subgame perfect equilibrium in which the monopolist generates a large profit when $b > 0$ and $\Delta > 0$ are small. The equilibrium constructed in Section 4.2 can serve as a credible threat to force the monopolist to follow a designated outcome path. Following the same idea as in Ausubel and Deneckere (1989), we can obtain the folk theorem if $\Delta \to 0$ and then $b \to 0$. In particular, we can sustain a subgame perfect equilibrium in which the monopolist generates an expected profit close to the static monopoly profit.

Proposition 5.1. *$\forall \epsilon > 0$, $\exists \overline{b} > 0$ and $\overline{y} > 1$ such that $\forall b \in (0, \overline{b})$ and $\forall y \in (1, \overline{y}]$ $\exists \overline{q}_f$ such that $\forall q_f \in [\overline{q}_f, 1)$ $\exists \overline{\Delta} > 0$ such that $\forall \Delta \in (0, \overline{\Delta})$, $\exists$ a subgame perfect equilibrium in which the equilibrium payoff $\mathcal{W}_s(\Delta)$ satisfies*

$$\mathcal{W}_s^m - \epsilon \leq \mathcal{W}_s(\Delta) \leq \mathcal{W}_s^m$$

where $\mathcal{W}_s^m$ is the static monopoly profit.

Proof. Apply Ausubel and Deneckere (1989). □

Following Ausubel and Deneckere (1989), we differentiate two kinds of subgame perfect equilibria: *reservation price equilibria* as defined in Definition 2.1, and *reputational equilibria*, where any deviation by the monopolist triggers a punishment phase as the continuation game is played according to the equilibrium constructed in Proposition 4.3. The key idea of Ausubel and Deneckere (1989) is to use the reputational equilibria to sustain an expected payoff close to the static monopoly profit as $\Delta \to 0$. As it is stated, Proposition 5.1 does not tell us whether the reputational effect or the slight decay is the key for the monopolist to generate a large profit. To crystallize the impact of a slight decay, we need to construct a reservation price equilibrium with a large profit in the perishable problem. The main goal of this section is to obtain Proposition 5.1 using only reservation price equilibria.

As in Section 4, we start with a simple artificial game to explore the key properties of the equilibrium we shall construct. Then, we construct a reservation price equilibrium, which generates expected profit close to the static monopoly profit for small $b > 0$.

5.1. **Another Artificial Game.**

The monopolist uses "burn off first and sell later" strategy as a way to influence the consumer's belief about the future prices offered by the monopolist. He is able to charge a higher price to high valuation customers because they believe a more reasonable offer is too far into the future. However, this strategy has an obvious downside: the monopolist has to delay the realization of profit. The monopolist has to balance the benefit of delaying and burning the available stock against the cost of delaying the profit.

To explore the tension between these two strategic motivations, let us examine a slightly more elaborate version of Section 4.1 where the monopolist can only delay the beginning of the game. Instead, let us allow the monopolist to choose a time interval with length $\tau > 0$ during which he chooses to burn the stock at the instant rate of e^{-b}. Thus, the sales can occur twice, immediately before and after the τ time break. Let (q_1, q_2) represent the total amount of goods delivered after each sales period, and (p_1, p_2) be the respective delivery prices. That is, at the beginning of the game, the monopolist charges p_1 to serve q_1, and then, takes a break for τ time. After the break, he charges p_2 to serve $q_2 - q_1$ additional consumers. As in Section 4.1, the initial quantity of the goods is y. All other parameters of the model remain the same as in Section 4.1.

We calculate the optimal solution through backward induction. Suppose that q_1 consumers have been served. Then, $y - q_1$ is available, and the residual demand curve is $\mathsf{D}(q_1, q_f)$. Throughout this example, we choose both $y > 1$ and $q_f < 1$ sufficiently close to 1, and $b > 0$ sufficiently small. Invoking the same logic as in Section 4.1, we have

$$q_2 - q_1 = (1 - q_1)\frac{r + b}{r + 2b}$$

and the monopolist has to delay the offer p_2 by τ in order to satisfy the market clearing condition:

$$e^{-b\tau}(y - q_1) = q_2 - q_1 = (1 - q_1)\frac{r + b}{r + 2b} \tag{5.16}$$

which implies that

$$p_2 = (1 - q_1)\frac{b}{r + 2b}.$$

Let $\tau(q_1)$ be the solution for (5.16). Note that

$$\tau'(q_1) > 0.$$

In order to make consumer q_1 indifferent between p_1 and p_2,

$$(1 - q_1) - p_1 = e^{-r\tau(q_1)}\left((1 - q_1) - (1 - q_1)\frac{b}{r + 2b}\right)$$

which implies that

$$p_1 = (1 - q_1)\left[1 - e^{-r\tau(q_1)}\frac{r + b}{r + 2b}\right].$$

Hence, the profit from selling q_1 in the first round can be written as

$$\begin{aligned} V(q_1) &= q_1(1 - q_1)\left[1 - e^{-r\tau(q_1)}\frac{r + b}{r + 2b}\right] + e^{-r\tau(q_1)}(1 - q_1)^2\frac{(r + b)b}{(r + 2b)^2} \\ &= (1 - q_1)\left(e^{-r\tau(q_1)}\frac{(r + b)b}{(r + 2b)^2} + q_1\left[1 - e^{-r\tau(q_1)}\frac{(r + b)(r + 3b)}{(r + 2b)^2}\right]\right). \end{aligned}$$

Note that as $b \to 0$, $V(q_1)$ converges uniformly to $(1-q_1)q_1$ over $q_1 \in [0, q_f]$. A simple calculation shows

$$\begin{aligned} V'(q_1) &= (1-2q_1)\left(1 - e^{-r\tau(q_1)}\frac{(r+b)(r+3b)}{(r+2b)^2}\right) \\ &\quad -e^{-r\tau(q_1)}\frac{(r+b)b}{(r+2b)^2} - \left[\frac{(r+b)b}{(r+2b)^2} - q_1\frac{(r+b)(r+3b)}{(r+2b)^2}\right] re^{-r\tau(q_1)}\tau'(q_1). \end{aligned}$$

As $\tau(q_1)$ is determined by (5.16), $\tau(q_1) \to \infty$ as $b \to 0$, as long as $y > 1$. Thus, the first term in the second line vanishes as $b \to 0$. To show that the second term in the second line also vanishes, recall (5.16). Thus,

$$e^{-r\tau(q_1)}\tau'(q_1) = \frac{\tilde{\omega}e^{-\frac{r\omega}{b}}}{b}$$

where

$$\omega = -\log\frac{(1-q_1)(r+b)}{(y-q_1)(r+2b)} > 0$$

and

$$\tilde{\omega} = \frac{(r+b) - e^{-b\tau(q_1)}(r+2b)}{b(1-q_1)}.$$

Since $y > 1$,

$$\lim_{b \to 0} e^{-r\tau(q_1)}\tau'(q_1) = 0$$

which implies that the second line vanishes as $b \to 0$. Thus,

$$\lim_{b \to 0} V'(q_1) = 1 - 2q_1,$$

and the delivery price, p_1, converges to 0.5, which generates the static monopoly profit. The slow rate of decay combined with a negligible profit from the continuation game makes it credible for the monopolist to delay an acceptable offer for an extremely long period.

5.2. **Reservation Price Equilibrium with Large Profit.** The key feature of the equilibrium constructed in Section 4.2 is that the monopolist can credibly delay to make an acceptable offer, when the expected profit from accelerating the sale is small. Following the same logic as in Section 4.2, we can construct another reservation price equilibrium, which generates the static monopoly profit as $\Delta \to 0$ as illustrated in Section 5.1.

Imagine an equilibrium that consists of two phases. In each phase, the monopolist is making a series of acceptable offers, denoted as $p_{1,1}$ and $\{p_{2,t}\}_{t=1}^{T_f}$ where the subscript represents the phase and the period within each phase. And, T_1 represents the number of periods during which the monopolist is making unacceptable offers. The first phase consists of a single offer, which is accepted by $\overline{q}_1$ consumers. After $\overline{q}_1$ consumers are served, the continuation game is played according to the "burn off first and sell later" equilibrium constructed in Section 4.2: the monopolist makes T_1 unacceptable offers and then makes a series of acceptable offers for T_f rounds to clear the market. We choose an optimal $\overline{q}_1$ that maximizes the expected discounted profit among all equilibria that have the same two phase structure as described above.

A minor complication is that the "burn off first and sell later" equilibrium exists only when the size of the residual demand is sufficiently large. Otherwise, the delay tactic by burning off the existing supply is not an optimal strategy. This is the reason why we may need a randomized strategy. In order to simplify the characterization of the optimal $\overline{q}_1$, however, let us assume for a moment that the continuation game strategy in the second phase is the "burn off first and sell later" equilibrium, because the analysis of the remaining case follows the same reasoning.

Let $V(0, q_f, y, \overline{q}_1)$ be the expected payoff of the monopolist if the demand curve is $\mathsf{D}(0, q_f)$, the supply is y and his initial offer is accepted by $\overline{q}_1$ portion of consumers, who expect that following the initial offer $p_{1,1}$, the monopolist will follow the delay strategy characterized in Section 4 associated with the residual demand $\mathsf{D}(\overline{q}_1, q_f)$ and the supply $e^{-b\Delta}(y - \overline{q}_1)$. Following the same notational convention as in (4.10), we can write

$$V(0, q_f, y, \overline{q}_1) = \max_{T_1, \mathbf{q} \in \mathbf{Q}} \overline{q}_1 p_{1,1} + e^{-r\Delta T_1} \sum_{t=1}^{T_f} p_t (q_t - q_{t-1}) e^{-r\Delta(t-1)} \tag{5.17}$$

subject to

$$(1 - q_t) - p_{2,t} = e^{-r\Delta}((1 - q_t) - p_{2,t+1}) \qquad \forall t \geq 1 \tag{5.18}$$

$$(1 - \overline{q}_1) - p_{1,1} = e^{-r\Delta T_1}((1 - \overline{q}_1) - p_{T_1+1}) \tag{5.19}$$

$$p_{T_f} = 1 - q_{T_f} \tag{5.20}$$

$$e^{-b\Delta(T_f - T_1 - 1)} \left(e^{-b\Delta T_1}(y - \overline{q}_1) - \sum_{t=1}^{T_f - T_1 - 1} e^{b\Delta t}(q_t - q_{t-1}) \right) \geq 0 \tag{5.21}$$

$$e^{-b\Delta(T_f - T_1 - 1)} \left(e^{-b\Delta T_1}(y - \overline{q}_1) - \sum_{t=1}^{T_f - T_1 - 1} e^{b\Delta t}(q_t - q_{t-1}) \right) (q_{T_f} - q_f) = 0, \tag{5.22}$$

where T_f is the number of rounds needed to clear the market, after the monopolist makes the first acceptable offer. In this case, T_f is precisely the round when the market clearing offer is made, because the monopolist's initial offer $p_{1,1}$ is accepted by a positive portion of consumers. Note that $p_{1,1}$ is completely determined by $\overline{q}_1$ and (5.19).

Let

$$\mathcal{W}_s(\Delta) = \max_{\overline{q}_1 \in [0, q_f]} V(0, q_f, y, \overline{q}_1)$$

and denote the optimal value of $\overline{q}_1$ as $\overline{q}_1^e$. Let $(\overline{q}_1^e, \mathbf{q}, T_1)$ be an optimal solution. We can show that $(\overline{q}_1^e, \mathbf{q}, T_1)$ can be sustained by a reservation price equilibrium.

Proposition 5.2. *$\exists \overline{b} > 0$ and $\overline{y} > 1$ such that $\forall b \in (0, \overline{b})$ and $y \in (1, \overline{y}]$ $\exists \overline{q}_f$ such that $\forall q_f \in [\overline{q}_f, 1)$, $(\overline{q}_1^e, \mathbf{q}, T_1)$ can be sustained by a reservation price equilibrium.*

Proof. See Appendix B. □

From the analysis of Section 4.1, we know that

$$\lim_{b \to 0} \lim_{\Delta \to 0} \Delta T_1 = \infty.$$

Hence,

$$(1 - \overline{q}_1) - p_{1,1} \to 0.$$

In particular, if $\overline{q}_1 = 0.5$, the resulting expected payoff converges to the static monopoly profit. Thus, if the monopolist chooses $\overline{q}_1$ optimally, the resulting profit $\mathcal{W}_s(\Delta)$ must converge to the static monopoly profit, which is the largest profit among all incentive compatible mechanism.

Proposition 5.3. *$\forall\epsilon > 0$ $\exists\overline{b} > 0$ and $\overline{y} > 1$ such that $\forall b \in (0, \overline{b})$ and $y \in (1, \overline{y}]$ $\exists\overline{q}_f$ such that $\forall q_f \in [\overline{q}_f, 1)$ $\exists\overline{\Delta} > 0$ such that $\forall\Delta \in (0, \overline{\Delta})$ $\exists$ a reservation price equilibrium in which the monopolist's expected profit is $\mathcal{W}_s(\Delta)$ such that $\mathcal{W}_s^m \leq \mathcal{W}_s(\Delta) + \epsilon$, where $\mathcal{W}_s^m$ is the static monopoly profit.*

6. Concluding Remarks

6.1. **General Demand Curve.** This paper has focused on linear market demand because the classic problem provides a crisp benchmark against which we can compare the outcome of the perishable problem to highlight the impact of slow decay. The idea of constructing a subgame perfect equilibrium can be carried over to the case of a general demand curve, in which the price is a strictly decreasing continuous function of the quantity. However, the actual construction of a subgame perfect equilibrium is much more involved than the case of linear demand, because we need to rely on mixed strategies extensively.

To show that the main idea of construction applies for general demand, and also to illustrate the source of complications, we examine in Appendix C the game with the demand function of the following form:

$$p = \begin{cases} 3 & \text{if} \quad q \leq x \\ 1 & \text{if} \quad x \leq q \leq x + 0.5 \\ 0 & \text{otherwise} \end{cases}$$

where $0 < x \leq 0.5$.

This is essentially the same example analyzed in Gul, Sonnenschein, and Wilson (1986), which is designed to show the need for a mixed strategy off the equilibrium path. We choose x so that the monopolist's optimal pricing sequence is $p_1 = 3 - 2\delta$ and $p_2 = 1$, which clears the market.

We construct a subgame perfect equilibrium which features "burn off first and sell later". The equilibrium requires an extensive use of a randomized strategy off the equilibrium path, but retains the key elements of the equilibrium constructed for the case of linear demand.

6.2. **Increasing Demand.** The strategic impact of decay arises from the fact that the excess demand for the goods increases as fewer goods become available. One can apply the same logic of the perishable problem to the case where demand is expanding. Sobel (1991) investigates the dynamic sales problem with entry of consumers in the market. Because the goods are sold to the high valuation consumers, the remaining consumers have lower reservation values and the residual demand curve becomes more elastic. As a result, the seller offers a low price in order to clear the market occasionally. We expect a similar dynamics. However, we also expect that, by burning off existing stock, the monopolist may not serve some low valuation consumers. This results in considerably delay in offering a sales price to clear the market. Formal analysis is left as a future research project.

6.3. **Endogenous Stock.** Perishable durable goods such as vaccine and cement are placed in storage until they are sold. Because delivery to the warehouse is made at different times, the inventory consists of goods with different expiration dates. The distribution of expiration dates determines the rate of decay. The present model provides insight about how sales can evolve for a given distribution of expiration dates, as we assume that the initial available stock is exogenous. We have yet to investigate how the pricing rule changes when the monopolist can control the distribution of expiration dates.

Appendix A. Proof of Proposition 4.2

We have to calculate the optimal strategy of the monopolist for all feasible configurations of $\mathsf{D}(q_0, q_f)$ and y. However, we can exploit the linearity of the demand curve to simplify the characterization substantially.

Lemma A.1. *Suppose that $\mathbf{p} = \{p_t\}$ and $\mathbf{q} = \{q_t\}$ are the optimal pricing and the quantity sequences of the constrained optimization problem (4.10) associated with $\mathsf{D}(0, q_f)$ and y, and it takes T_f periods to clear the market. If the demand curve is given by $\mathsf{D}(1-\alpha, 1-\alpha+\alpha q_f)$ and the initial stock is αy, then $\alpha\mathbf{p}$ and $\alpha\mathbf{q}$ are the solution to the constrained optimization problem, and trading is completed in exactly T_f periods. This relation holds $\forall b \geq 0$ (both for the perishable and for the classic problems).*

Proof. The proof follows from the fact that the objective function and the constraints are linear functions of $q_t - q_{t-1}$. □

For a moment, let us consider the optimization problem of the classic problem. Given residual demand $\mathsf{D}(\underline{q}, \overline{q})$ ($\underline{q} < \overline{q} \leq q_f$), let $\{q_t\}_{t=1}^{T_f}$ be an optimal solution for the monopolist's profit maximization problem. Define

$$y_f(\underline{q}, \overline{q}) = \sum_{t=1}^{T_f} (q_t - q_{t-1})\beta^{-t} \tag{A.23}$$

If the monopolist begins to offer an acceptable offer immediately to meet the residual demand $\mathsf{D}(\underline{q}, \overline{q})$, the available stock must be $y_f(\underline{q}, \overline{q})$.

Lemma A.2. (1) *$y_f(\underline{q}, \overline{q})$ is a decreasing function of $\underline{q}$. In particular,*

$$-\frac{y_f(\underline{q}, \overline{q}) - y_f(\underline{q}', \overline{q})}{\underline{q} - \underline{q}'} \geq \frac{1}{\beta}. \tag{A.24}$$

(2) *$y_f(\underline{q}, \overline{q})$ is a strictly increasing function of $\overline{q}$.*

Proof. Fix $\underline{q}' < \underline{q}$. In the classic problem, the optimal solution is constructed backward. Thus, the optimal solution for $\mathsf{D}(\underline{q}', \overline{q})$ is to follow an optimal pricing rule up to $\underline{q}$, and then to follow the optimal pricing sequence associated to $\mathsf{D}(\underline{q}, \overline{q})$. Let $\{q_t\}_{t=1}^{T_f}$ be the optimal solution associated with $\mathsf{D}(\underline{q}', \overline{q})$. Let T' be the periods needed to serve $\underline{q} - \overline{q}'$ consumers. Then,

$$\begin{aligned} y_f(\underline{q}', \overline{q}) &= \sum_{t=1}^{T'} (q_t - q_{t-1})\beta^{-t} + \sum_{t=T'+1}^{T_f} (q_t - q_{t-1})\beta^{-t} \\ &\geq (\underline{q} - \underline{q}')\beta^{-1} + \sum_{t=2}^{T_f} (q_t - q_{t-1})\beta^{-t} \\ &\geq (\underline{q} - \underline{q}')\beta^{-1} + y_f(\underline{q}, \overline{q}) \end{aligned}$$

The first inequality follows from the observation that serving $\underline{q} - \underline{q}'$ instantaneously is the quickest way to reach $\underline{q}$. After arranging the terms, we have

$$-\frac{y_f(\underline{q}, \overline{q}) - y_f(\underline{q}', \overline{q})}{\underline{q} - \underline{q}'} \geq \frac{1}{\beta} + \frac{y_f(\underline{q}, \overline{q})\beta^{-1}}{\underline{q} - \underline{q}'}(\beta^{-1} - 1) \geq \frac{1}{\beta} \tag{A.25}$$

as desired.

To prove the second part, we can assume without loss of generality that $\underline{q} = 0$. From Lemma A.1, we know

$$y_f(1-\alpha, (1-\alpha) + \alpha\overline{q}) = \alpha y_f(0, \overline{q}).$$

Since $\overline{q} < 1$, it suffices to show

$$y_f(0, (1-\alpha) + \alpha\overline{q}) > y_f(0, \overline{q}).$$

Since $\overline{q}$ is arbitrary, it suffices to show that the strict inequality holds for α sufficiently close to 1. Recall (A.25). By substituting $\underline{q}$ by $1-\alpha$ and $\overline{q}$ by $1-\alpha+\alpha\overline{q}$, we have

$$\begin{aligned} y_f(0, 1-\alpha+\alpha\overline{q}) &\geq \frac{1-\alpha}{\beta} + \frac{y_f(1-\alpha, 1-\alpha+\alpha\overline{q})}{\beta} \\ &= \frac{1-\alpha}{\beta} + \frac{\alpha y_f(0,\overline{q})}{\beta} \end{aligned}$$

where the equality follows from Lemma A.1. After arranging terms, we have

$$y_f(0, 1-\alpha+\alpha\overline{q}) - y_f(0,\overline{q}) \geq \frac{1-\alpha}{\beta} + \frac{\alpha-\beta}{\beta} y_f(0,\overline{q}) > 0$$

if $\alpha \in (\beta, 1)$. □

Because of decay, the monopolist initially needs more than the actual amount that will be delivered eventually:

$$q \leq y_f(0,q).$$

If $y_f(0,q) \geq y$, then the existing stock is too small to serve $\mathsf{D}(0,q)$. By (A.24), $y_f(q_0,q) - \beta(y-q_0)$ is a decreasing function of q_0. Moreover, $y_f(0,q) \geq y$ by assumption, and $y_f(q,q) = 0 < \beta(y-q)$, since $y > q$. Since $y_f(q_0,q)$ is continuous with respect to q_0, we have

$$y_f(q_0,q) = \beta(y-q_0)$$

for some $q_0 \in [0,q]$.

The constrained optimal pricing rule is thus an acceleration strategy defined as follows.

Definition A.3. *An acceleration strategy for $\mathsf{D}(0,q)$ is an outcome path in which the monopolist serves q_0 consumers immediately, and then follows the optimal pricing sequence associated with $\mathsf{D}(q_0,q)$. The initial offer p' is determined according to*

$$1 - q_0 - p' = \delta((1-q_0) - p_1)$$

where p_1 is the initial offer from the optimal pricing sequence associated with $\mathsf{D}(q_0,q)$.

If $y_f(0,q) \leq y$, then the existing stock is too large to credibly serve q, because the terminal condition (4.12) does not hold for y. The monopolist follows another outcome path, a delay strategy, defined as follows.

Definition A.4. *A delay strategy is an outcome path in which the monopolist makes unacceptable offers for T_1 periods, where*

$$T_1(0,q,y) = \inf\left\{T : e^{-b\Delta T} y \leq y_f(0,q)\right\}$$

and then, follows the acceleration strategy.

A natural state variable is the residual demand $\mathsf{D}(q_0,q_f)$ and the available stock at the time when the monopolist makes the decision. By a state, we mean a triple (q_0,q_f,y) representing residual demand and the available stock.

Let $\overline{q}^*(0,q_f,y)$ be the total amount of goods served in an optimal solution of (4.10) where the state is $(0,q_f,y)$. If $\overline{q}^*(0,q_f,y) = q_f$, then the associated optimal pricing sequence is precisely the optimal pricing sequence from the classic problem, because constraint (4.13) is not binding. Otherwise, constraint (4.13) is binding, and some consumers are not served as the available goods are burned off, and therefore, the optimal strategy is a delay strategy.

Based on the analysis of the optimal strategy under state $(0,q_f,y)$, we have a "rough" characterization of the optimal strategy for an arbitrary state (q_0,q_f,y) and $\overline{q}^*(q_0,q_f,y)$ which is the counterpart of $\overline{q}^*(0,q_f,y)$ for state (q_0,q_f,y):

- if $\overline{q}^*(q_0,q_f,y) \geq \min(q_f,y)$, the monopolist follows the acceleration strategy, and
- if $\overline{q}^*(q_0,q_f,y) < \min(q_f,y)$, then the monopolist delays $T_1(q_0,\overline{q}^*(q_0,q_f,y),y)$ periods before making acceptable offers. After making $T_1(q_0,\overline{q}^*(q_0,q_f,y),y)$ unacceptable offers, the monopolist follows the acceleration strategy associated with state $(q_0,q_f,e^{-b\Delta T_1(q_0,\overline{q}^*(q_0,q_f,y),y)}y)$.

It is only a rough characterization, because we have yet to identify how many consumers will accept an offer p_1' which is not an equilibrium offer. We shall focus the analysis on the deviation from the first offer in the equilibrium, because the general case follows from the same logic.

We need to consider two separate cases depending upon whether the initial offer is acceptable (i.e., the monopolist follows an acceleration strategy), or the initial offer is unacceptable (i.e., the monopolist follows a delay strategy).

A.1. p_1 **is an acceptable equilibrium offer.** Fix $p_1' \neq p_1$. We only examine the case where $p_1' < p_1$, because the other case follows from the symmetric logic. If the acceptable strategy does not bind (4.13), then the complementary slackness condition implies that

$$q_f = \overline{q}^*(0, q_f, y).$$

Since we are considering an acceleration strategy illustrated in Definition A.3, $T_f(\cdot)$ is precisely the total number of periods when the market is open. In this case, the equilibrium strategy off the equilibrium path is identical to that in the classic problem. Because the unique subgame perfect equilibrium in the classic problem is a reservation price equilibrium, the acceleration strategy can be sustained by a reservation price equilibrium. By the nature of the reservation price equilibrium for the classic problem, a lower than equilibrium offer price increases sales in that period. As a result, condition (4.13) is not binding in any continuation game. That is why we can use the same reservation price equilibrium strategy of the classic problem, acting as if the good does not perish.

On the other hand, if the acceptable strategy binds condition (4.13) so that

$$q_f > \overline{q}^*(0, q_f, y),$$

then the equilibrium strategy off the equilibrium path differs from that of the classic problem. Yet, we can show that a lower than an equilibrium price offer always increases sales in that period.

In this case, condition (4.13) holds with equality, and the market must be cleared in the sense that the monopolist sells all available stock, even though some consumers are not served. Since p_1 is an acceptable offer, all ensuing offers from the monopolist are also acceptable for some consumers. Thus, after q_1 consumers are served,

$$y_f(q_1, \overline{q}^*(q_1, q_f, y)) = \beta(y - q_1) \tag{A.26}$$

must hold, where the left hand side is the amount of goods needed to serve the remaining consumers in the continuation game following p_1, while the right hand side is the amount of goods available at the beginning of the second round.

Fix $p_1' < p_1$, and let q_1' and q_1 be the mass of consumers who accept p_1' and p_1, respectively. We claim that $q_1' \geq q_1$. By Lemma 2.3, there exists $q' > \overline{q}^*(0, q_f, y)$ such that the consumer expects the total amount of delivery is q' and forms the acceptance rule accordingly. By the definition of y_f,

$$y_f(q_1', q') = \beta(y - q_1') \tag{A.27}$$

must hold, which is analogous to (A.26). However, if both (A.26) and (A.27) hold, then

$$y_f(q_1', q') - y_f(q_1, \overline{q}^*(q_1, q_f, y)) = \beta(q_1 - q_1').$$

After arranging the terms, we have

$$\frac{y_f(q_1', q') - y_f(q_1, q')}{q_1 - q_1'} = \beta - \frac{y_f(q_1, q') - y_f(q_1, \overline{q}^*(q_1, q_f, y))}{q_1 - q_1'}.$$

The second term in the right hand side is non-negative, because $q' \geq \overline{q}^*(q_1, q_f, y)$. Thus,

$$\frac{y_f(q_1', q') - y_f(q_1, q')}{q_1 - q_1'} \leq \beta$$

which implies that $q_1' \geq q_1$. Otherwise, it contradicts to (A.24). This contradiction proves that $q_1' \geq q_1$, thus proving that the constructed equilibrium is a reservation price equilibrium.

A.2. p_1 **is an unacceptable equilibrium offer.** The construction follows almost the same idea. Without loss of generality, set $q_0 = 0$. While there are many unacceptable offers, let us streamline the construction by focusing on a series of "lowest" unacceptable offers that makes it indifferent for the highest reservation value consumer in the market to accept and reject the present offer.

Suppose that the monopolist makes T_1 unacceptable offers, before making the first acceptable offer p_{T_1+1}. Define p_t $(t \leq T_1)$ implicitly as

$$1 - p_t = \delta^{T_1-t+1}(1 - p_{T_1+1})$$

or equivalently as

$$p_t = (1 - \delta^{T_1-t+1})(1 - q_0) + \delta^{T_1-t+1} p_{T_1+1}.$$

In particular,

$$p_1 = (1 - \delta^{T_1})(1 - q_0) + \delta^{T_1} p_{T_1+1}.$$

Fix $p_1' \neq p_1$. Let us focus on the case where $p_1' < p_1$, since the case of $p_1' > p_1$ follows from the symmetric argument. We need to find an optimal strategy with the additional constraint that the initial offer is p_1'. By Lemma 2.3, we know that the initial acceptable offer is a continuous function of the terminal offer. Since the initial unacceptable offer is a continuous function of the first acceptable offer, it is also a continuous function of the terminal offer. Recall that $\overline{q}^*(0, q_f, y)$ is the equilibrium quantity delivered to the consumers. For each $\overline{q} \geq \overline{q}^*(0, q_f, y)$, we construct an optimal pricing rule with the terminal condition that $p_{T_f} = 1 - \overline{q}$. Let $p_1^*(\overline{q})$ be the first offer (which may be unacceptable) in the optimal pricing rule that terminates with $p_{T_f} = 1 - \overline{q}$.

Since $p_1^*(\overline{q})$ is a continuous function of $\overline{q}$, $\forall p_1' < p_1$, $\exists \overline{q} > \overline{q}^*(0, q_f, y)$ such that $p_1' = p_1^*(\overline{q})$. If $p_1' = 1 - q_f$, then every consumer must accept the offer, because the monopolist will never charge a price lower than $1 - q_f$. Thus, there exists $\overline{q}'$ such that $p_1' = p^*(\overline{q}') \geq 1 - q_f$ is an acceptable offer. By the construction of the strategy off the equilibrium path from the acceptable offer, we know that the mass of consumers who accepts $p_1'' \leq p_1'$ does not decrease.

Appendix B. Proof of Proposition 5.2

B.1. **Preliminaries.** First, we need to identify the condition under which the "burn off first and sell later" equilibrium exists in the second phase. Recall the definitions of an accelerations strategy (Definition A.3 and a delay strategy (Definition A.4).

Lemma B.1. *$\exists q_f' < 1$ and $y' > 1$ such that $\forall q_f \in (q_f', 1)$ and $y \in (1, y')$ $\exists \overline{\Delta} > 0$ and $\overline{b} > 0$ such that $\forall \Delta \in (0, \overline{\Delta})$ and $b \in (0, \overline{b})$ $\exists q^*$ such that if $q_0 > q^*$, then the acceleration strategy can be optimal, but the delay strategy cannot be optimal. If $q_0 < q^*$, then the delay strategy can be optimal for residual demand $\mathsf{D}(q_0, q_f)$ with available stock y.*

Proof. Since the payoff from the two strategies changes continuously with respect to $\Delta > 0$, let us consider the limit case examined in Section 4.1. Fix a residual demand $\mathsf{D}(q_0, q_f)$ and the available stock y. From the acceleration strategy, the monopolist obtains $q_f(1 - q_f)$ instantaneously. On the other hand, from the delay strategy, he obtains

$$e^{-r\tau(q_0)}(1 - q_0)^2 \frac{(r + b)b}{(r + 2b)^2}$$

where $\tau(q_0)$ is defined implicitly by (5.16) with q_1 replaced by q_0. Choose $q_f' < 1$ and $y' > 1$ sufficiently close to 1 so that

$$q_f(1 - q_f) < e^{-r\tau(q_0)}(1 - q_0)^2 \frac{(r + b)b}{(r + 2b)^2}.$$

We know that $\tau'(q_0) > 0$, and also, for a given q_0, $\tau(q_0)$ increases without a bound as $b \to 0$. Thus, for any sufficiently small $b > 0$, we can find a critical q_0 where the above strict inequality holds with equality. This is q^*. By the continuity of the expected payoff with respect to $\Delta > 0$, we can repeat the same reasoning for a small $\Delta > 0$ to find q^*. ◻

Remark B.2. *We can state the same result in terms of y for a fixed demand $\mathsf{D}(0, q_f)$. That is, if y is sufficiently large, then it takes exceedingly long period to burn off the existing stock. It makes no sense to use the delay strategy, and only the acceleration can be an optimal strategy.*

Let us consider the initial demand $\mathsf{D}(0, q_f)$ and the initial stock y. After $\overline{q}_1$ consumers are served, the continuation game is played with the residual demand curve $\mathsf{D}(\overline{q}_1, q_f)$ and available stock $\beta(y - \overline{q}_1)$. Since we choose $y > 1$, and $q_f < 1$, we can invoke Lemma B.1 to identify whether the continuation game can sustain the delay strategy as a Nash equilibrium.

Corollary B.3. *$\exists q'_f < 1$ and $y' > 1$ such that $\forall q_f \in (q'_f, 1)$ and $y \in (1, y')$ $\exists \overline{\Delta} > 0$ and $\overline{b} > 0$ such that $\forall \Delta \in (0, \overline{\Delta})$ and $b \in (0, \overline{b})$ $\exists \overline{q}_1^*$ such that the continuation game after $\overline{q}_1$ consumers are served can sustain the delay strategy as a Nash equilibrium if and only if $\overline{q}_1 \leq \overline{q}_1^*$.*

Proof. Note that $y > 1$ and $q_f < 1$. It is clear that $\inf_{\overline{q}_1} e^{-b\Delta}(y - \overline{q}_1) - (q_f - \overline{q}_1) > 0$ as long as $\Delta > 0$ is sufficiently small. Thus, $\forall \overline{q}_1 \leq q_f$, the amount of delay, $\tau(\overline{q}_1)$, defined by (5.16) increases without a bound as $b \to 0$. The conclusion follows from the same reasoning as the proof of Lemma B.1. □

B.2. **Construction of an Equilibrium.** Choose the parameters according to Lemma B.1 so that $q^* \in [0, q_f]$ exists. Fix the outcome path associated the optimal value $\overline{q}_1^e$. We need to construct the strategy off the equilibrium path. We focus on the initial offer because we already know that the second phase can be sustained by a reservation price equilibrium if $\overline{q}_1^e \leq q^*$. If we choose the parameters according to Lemma B.1, then $\overline{q}_1^e \leq q^*$ for any sufficiently small $\Delta > 0$.

Recall our convention of state: the pair of residual demand $\mathsf{D}(q_0, q_f)$ and the existing supply y. For simplicity, we treat (q_0, q_f, y) as a state. Fix $p'_1 \neq p_{1,1}$. We focus on the case $p'_1 < p_{1,1}$, because the other case follows from symmetric reasoning. Since $p_{1,1}$ must satisfy (5.19), $\forall p'_1 < p_{1,1}$ $\exists \overline{q}'_1$ and a delay strategy in $\mathsf{D}(\overline{q}'_1, q_f)$ with the available stock $\beta(y - \overline{q}'_1)$ such that

$$1 - \overline{q}'_1 - p'_1 = \delta^{T'_1}(1 - \overline{q}'_1 - p'_{2,1})$$

where T'_1 and $p'_{2,1}$ are the number of unacceptable offers and the unacceptable initial offer in the delay strategy associated with state $(\overline{q}'_1, q_f, \beta(y - \overline{q}'_1))$. Since the initial offer of the delay strategy associated with state $(\overline{q}'_1, q_f, \beta(y - \overline{q}'_1))$ is a continuous function of $\overline{q}'_1$, we can choose

$$\overline{q}_1(p'_1) = \sup\{\overline{q}'_1 : \text{there is a delay strategy associated with } (\overline{q}'_1, q_f, \beta(y - \overline{q}'_1)) \text{ satisfying (5.19)}\}. \tag{B.28}$$

We know the mapping $p_{1,1} \mapsto \overline{q}_1(p_{1,1})$ may not be invertible but the "sup" operator in the definition ensure that $\overline{q}_1(p_{1,1})$ is a strictly decreasing function of $p_{1,1}$. Define

$$p_1^* = \overline{q}_1^{-1}(q^*)$$

as the lowest possible price such that $q_1 = q^*$ and therefore, the ensuing game can be played by the delay strategy. If $q^* = \overline{q}_1(p_{1,1})$ holds at multiple values of $p_{1,1}$ (which is the case when $\overline{q}_1$ has the downward jump, then choose p_1^* as

$$p_1^* = \inf \overline{q}_1^{-1}(q^*)$$

Recall that if the initial offer is less than or equal to $1 - q_f$, then every consumer accepts of the offer, and therefore, the continuation game cannot be sustained by the delay strategy. Thus, $p_1^* > 1 - q_f$.

If $p'_1 \geq p_1^*$, then the continuation game is played by the delay strategy, as constructed for the equilibrium path. The only difference is to change q_1^e by q'_1 associated with the deviation p'_1.

If $p_{1,1} < p_1^*$, then the continuation game cannot be played by the delay strategy. Instead, the continuation game is played by the acceleration strategy. However, we need to do some work to "smooth" out the discontinuity created as the strategy in the ensuing game switches from the delay strategy to the acceleration strategy,

Consider the continuation game after q^* consumers are served. The residual demand is $\mathsf{D}(q^*, q_f)$ and the available stock is $\beta(y - q^*)$. By the definition of q^*, both the acceleration and the delay strategies are optimal. Let $p_{2,1}^a$ be the initial offer of the acceleration strategy, and $p_{2,1}^d$ be the initial offer of the delay strategy, which is offered after T_1 periods. Since $p'_1 < p_1^*$,

$$1 - q^* - p'_1 > \delta^{T_1}(1 - q^* - p_{2,1}^d).$$

If $p_1' > p^a_{2,1}$, then

$$1 - q^* - p^a_{2,1} > 1 - q^* - p_1'.$$

Choose $\alpha \in [0,1]$ such that

$$1 - q^* - p_1' = \alpha\delta^{T_1}(1 - q^* - p^d_{2,1}) + (1-\alpha)(1 - q^* - p^a_{2,1}).$$

Such α exists as long as $p_1' \in [p^a_{2,1}, p_1^*]$. That is, the consumers expect that the monopolist randomizes over two strategies so that the q^* consumer is indifferent between accepting and rejecting p_1'.

Since we choose Δ sufficiently small, and since $p^a_{2,1} \to 1 - q_f$, $p^a_{2,1} < p_1^*$. However, if $p_1' < p^a_{2,1}$, then the continuation game is played according to the acceleration strategy with probability 1.

Appendix C. When the Demand Function is a Step Function

Abusing notation slightly, let x be the mass of high value consumers and y be the amount of supply. A natural state variable is (x, y) which characterize the residual demand and the quantity of available goods. If $x \geq y$, then the monopolist can charge 3 and serve all remaining high valuation consumers, credibly excluding the low valuation consumers. The key decision is how long the monopolist has to wait before he can credibly charge 3.

Essentially, the monopolist has three options at (x, y).

- **Accelerating.** The monopolist can accelerate sales in one of two ways. The first way is to serve everyone in the market. His profit will be

$$y. \tag{C.29}$$

An alternative method is to charge $3 - 2\delta$ which is accepted by all high valuation buyers, and in the following round, charge 1 which is accepted by the remaining low valuation buyers. The average discounted profit is

$$(3 - 2\delta)x + \delta\beta(y - x). \tag{C.30}$$

Whether (C.29) or (C.30) is optimal depends upon the size of x.

- **Delaying.** Continue to charge 3 until the high valuation consumer concludes that the monopolist will not lower the price, which makes the high valuation buyer accept the offer. Let k be the first period that

$$\beta^{k-1} y \leq x$$

which implies that the monopolist can credibly charge 3 since high valuation buyer will accept the offer immediately. Thus, if it takes k rounds, the expected profit is

$$3\delta^{k-1}\min(\beta^{k-1}y, x). \tag{C.31}$$

In order to delineate the optimal action of the monopolist under (x, y), let us characterize the "indifference state" between the Coase conjecture type strategy and the last one involving delay. That is state (x, y) solving

$$\max\left(y, (3 - 2\delta)x + \delta\beta(y - x)\right) = 3\delta^{k-1}\min(\beta^{k-1}y, x). \tag{C.32}$$

assuming for a moment that k can take any positive real number.

Lemma C.1. *Suppose that k can be any non-negative real number.*

1. $\forall (x, y)$, $\exists k \geq 0$ *such that* (C.32) *holds.*
2. *If (x, y) satisfies* (C.32)*, then so does $(\lambda x, \lambda y)$ $\forall \lambda > 0$.*
3. *Define*

$$\mathcal{K} = \{k : \exists (x, y) \quad \textit{such that } \text{(C.32)} \textit{ holds.}\}.$$

Then, $\mathcal{K}$ is a compact and connected set and therefore,

$$\sup \mathcal{K} < \infty.$$

4. *For a fixed x, and $y' > y$. Let k and k' be the solutions associated with (x, y) and (x, y') in* (C.32)*, respectively. Then, $k' > k$.*

Proof. Define

$$g(k) = \max(y, (3-2\delta)x + \delta\beta(y-x)) - 3\delta^{k-1}\max(\beta^{k-1}y, x).$$

which is a continuous function of k. Note $g(0) < 0$ and $\lim_{k\to\infty} g(k) > 0$. Moreover, $g(k)$ is a strictly increasing function of k. Thus, there exists a unique k satisfying (C.32). Note that $g(k)$ is a linear function of (x, y) which implies the second statement. We know that the mapping $(x, y) \mapsto k$ satisfying (C.32) is continuous. Since (x, y) is contained in a compact set, $\mathcal{K}$ is compact, which implies the third statement. The last statement follows from the fact that the greater the existing level of stock is, the longer it takes to reach the area where $y \leq x$. This is depicted in Figure 2. □

For a fixed x, there is a one-to-one correspondence between (x, y) and the solution, k, from (C.32). For each k, define $\alpha(k) = y/x$ where (x, y) induces k as the solution of (C.32). From Lemma C.1,

$$\mathcal{U}(k) = \{(x, y) : k \text{ is the solution of } (C.32)\}$$

is a half line through the origin with slope $\alpha(k)$ which is a strictly decreasing function of k. The slope of $\alpha(k)$ can range from 1 to $+\infty$.

$\mathcal{U}(k)$ represents the collection of states that make the monopolist indifferent between the two options of accelerating and delaying *if the delay takes k periods*. However, there is no guarantee that k is self-fulfilled. Again, let us assume for another moment that k can take any positive real number. Given (x, y), we can find a unique $k > 0$ such that

$$\beta^{k-1}y = x$$

which is the first time that the monopolist can credibly charge 3 and the offer is accepted by the remaining high valuation buyers with probability 1. Define

$$\mathcal{V}(k) = \quad (x, y) : y = \frac{1}{\beta^{k-1}}x$$

as the collection of states which take k periods to reach the area

$$\{(x, y) : y \leq x\}$$

where the monopolist's offer of 3 is accepted with probability 1. Note that $\mathcal{V}(k)$ is a half-line passing through the origin. Its slope ranges from 1 to $+\infty$ and is a strictly increasing function of k.

Therefore, $\exists k^* > 0$ such that

$$\mathcal{V}(k^*) = \mathcal{U}(k^*). \tag{C.33}$$

This k^* has a special meaning in the sense that if $(x, y) \in \mathcal{U}(k^*)$, the monopolist expects that in k^* periods, his offer of 3 will be accepted with probability 1 and indeed, it takes k^* period before this event occurs.

Remark C.2. *If k can take only a positive integer value, the same analysis proves the existence of a positive integer k^* such that*

$$\frac{1}{\beta^{k^*}} \geq \alpha(k^*) \geq \frac{1}{\beta^{k^*-1}}. \tag{C.34}$$

According to the definition of $\alpha(k^*)$ and $\mathcal{U}(k^*)$, if $y > \alpha(k^*)x$, then the monopolist charges 1 or $3 - 2\delta$ to 3, depending upon the size of x. Unless $x > 0$ is too small, the monopolist immediately makes an offer $3 - 2\delta$ which is accepted by all high valuation seller whose mass is x, and offers 1 next period to clear the market. As depicted in Figure 2, the state moves along the 45 degree line passing through (x, y), because each consumer demands exactly one unit. On the other hand, if $x < y < \alpha(k^*)x$, then the monopolist refuses to make an acceptable offer. For analytic convenience, let us assume that the monopolist charges $3 + \epsilon$ for a small $\epsilon > 0$, which is rejected by all high valuation consumers. The available stock decays at the rate of β in each period. After k periods of rejected offers, suppose that $\beta^k y < x < \beta^{k-1}y$ holds. If the high valuation consumers reject $3 + \epsilon$, then $\beta^k y < x$ implies that from the next period, there is excess demand among high valuation consumers and the monopolist can charge 3. Thus, all high valuation consumers are willing to accept any offer up to 3. Knowing this, the monopolist charges 3, following k unacceptable offers.

Note that if Δ consumers purchase the good in this round with state (x, y), then the state at the end of this round is $(x - \Delta, y - \Delta)$. At the beginning of the next round, the state becomes $(x - \Delta, \beta(y - \Delta))$.

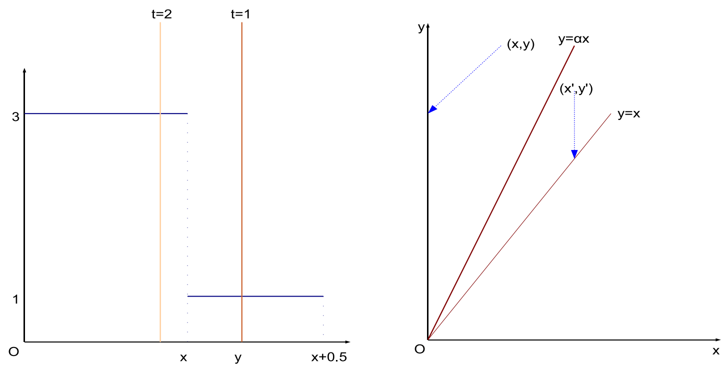

Figure 2: The left panel illustrates how the available stock decays in the case that the monopolist makes no sales in the first two rounds when $\Delta > 0$ is relatively large. The right panel depicts the area of (x, y) associated with the two different strategies when $\Delta > 0$ is small. The bold straight line is $y = \alpha x$. The monopolist makes an acceptable offer immediately if $y > \alpha x$. Note that if the monopoly makes sales, x and y decrease by the same amount, and (x, y) moves along the 45 degree line passing through the initial state. If all high valuation consumers are served, then x becomes 0. If $y' < \alpha x'$, then the state moved down vertically because no sales are made until the state hits $y = x$.

Consider the specific example of an initial state of $(0.5, y)$. If the initial state, $(0.5, y)$, is above $\mathcal{U}(k^*)$, then the construction of the actions off the equilibrium path follows the same idea as the weak stationary equilibrium in Gul, Sonnenschein, and Wilson (1986) with minor twist. We only describe the case where the monopolist charges $3 - 2\delta$ along the equilibrium path. Let $\mathcal{U}^*$ be the half line passing through the origin along which the monopolist is indifferent between charging 1 and $3 - 2\delta$. Let α^* be the slope of $\mathcal{U}^*$. One can easily show that

$$\alpha^* > \alpha(k^*) > 1.$$

If the initial equilibrium offer is $3 - 2\delta$, then the initial state is located between $\mathcal{U}^*$ and $\mathcal{U}(k^*)$.

If $p > 3 - 2\delta^2$, then no consumer accepts the offer, expecting that in the following period, the monopolist will charge $3 - 2\delta$. If $p < 3 - 2\delta$, then every consumer purchases the good. The state moves from $(0.5, y)$ to $(0, y - 0.5)$, which implies that the monopolist has some goods for future sale, because $y > 0.5$. In the next round following such a p, the monopolist charges 1 to serve all low valuation consumers.

If $3 - 2\delta < p < 3 - 2\delta^2$, we first locate a point along

$$y = \frac{\alpha^*}{\beta} x$$

that intersects with the 45 degree line passing through initial state $(0.5, y)$. Let $(0.5 - x^*, y - x^*)$ be such a point. Such a p is accepted by x^* portion of high valuation consumers who expect that in the following period, the monopolist randomizes between 1 and $3 - 2\delta$ with probability $1 - \lambda$ assigned to $3 - 2\delta$. so that

$$3 - p = \delta(3 - (\lambda \cdot 1 + (1 - \lambda)(3 - 2\delta))).$$

In the following round, $\beta(y - x^*)$ is available and the new state $(0.5 - x^*, \beta(y - x^*))$ is located along $\mathcal{U}^*$ where the monopolist is indeed indifferent between 1 and $3 - 2\delta$.

If state (x, y) is below $\mathcal{U}(k^*)$ but $y < x$, then the high valuation consumer accepts any offer $p < 3$. Finally, suppose that state (x, y) is below $\mathcal{U}(k^*)$ but $y > x$. For simplicity, let us assume that the monopolist is indifferent between charging 3 and $3 - 2\delta$ along $\mathcal{U}(k^*)$. The other case follows from the same logic, where the monopolist is indifferent between charging 3 and 1 along $\mathcal{U}(k^*)$.

The monopolist charges 3 in the equilibrium. If he charges $p > 3$, it is clearly optimal for the consumer to reject the offer with probability 1. If he charges $p \leq 3 - 2\delta$, then every high value consumer accepts the offer with probability 1, expecting that the monopolist will charge 1 in the following round. Indeed, after serving all high valuation consumers, the monopolist still has $\beta(y-x)$ amount of goods for sale in the next round. He charges 1 to serve some of the low valuation consumers.

Suppose that the monopolist charges $p \in (3-2\delta, 3)$. Recall that $\alpha(k^*) > 1$. Find a point along

$$y = \frac{\alpha(k^*)}{\beta} x$$

that intersects with the 45 degree line passing through the given state (x, y). Let $(x-x', y-y')$ be the intersection. Given p, x' portion of consumers accept the offer, expecting that the monopolist will randomize between 3 and $3-2\delta$ in the following round. Indeed, in the following round, the state is $(x-x', \beta(y-y'))$ which is located on $\mathcal{U}(\alpha^*)$, where the monopolist is indifferent between charging $3-\delta$ and 3.

This completes the construction of the equilibrium strategy. The construction process reveals that the monopolist cannot gain by further delaying the sales. The remaining step is to prove that the monopolist cannot benefit from accelerating sales. In particular, given the fact that the monopolist has to charge 3 which is not accepted by any buyer for a long time, it is not obvious whether or not a slight price cut can increase the profit of the monopolist.

To complete this part of the proof, let us fix state (x, y). If $y \le x$, then the equilibrium offer 3 is accepted with probability 1. Thus, it is obvious that the monopolist has no incentive to lower his price.

If $y > \frac{1}{\beta^{k^*}} x$, then the continuation game is played according to the subgame perfect equilibrium for the classic problem, because acceleration is optimal.

If $\frac{1}{\beta^{k^*}} x \le y < x$, the equilibrium dictates that the monopolists uses the delay strategy by continuously charging 3. This is rejected by the consumers until the available stock falls below x.

If $p' > 3$, the continuation game play is exactly the same as along the equilibrium path. It remains to construct the continuation game strategy following $p' < 3$ for each state (x, y) where $\frac{1}{\beta^{k^*}} x \le y < x$.

Given (x, y), define

$$\ell(x, y) = \min_{\ell = 1, 2, \ldots} \{\ell : \beta^{\ell} y \le x\}$$

as the first time when the existing stock falls below the size of the high valuation consumers. Let $\ell_0 = k^*$. Define $\mathcal{F}(\ell_0)$ as the collection of states (x, y) satisfying the following two conditions:

$$\exists \Delta, \text{ such that } \beta(y-\Delta) = \frac{1}{\beta^{\ell_0 - 1}}(x-\Delta) \tag{C.35}$$

$$3\Delta + \delta(3(y-\Delta)(\beta\delta)^{\ell_0}) \le (\beta\delta)^{\ell(x,y)} y. \tag{C.36}$$

Imagine that the monopolist charges a price $p' < 3$ which is very close to 3. If $(x, y) \in \mathcal{F}(\ell_0)$, then we can find a portion of high valuation consumers such that after Δ portion of high valuation consumers are served, the monopolist is indifferent between the acceleration and delay strategies, since $(x-\Delta, \beta(y-\Delta))$ is located where

$$\beta(y-\Delta) = \frac{1}{\beta^{\ell_0}}(x-\Delta)$$

according to condition (C.35). By deviating to $p' < 3$, the maximum payoff from the deviation is given by the left hand side of condition (C.36), which does not exceed the equilibrium payoff $(\beta\delta)^{\ell(x,y)} y$.

It is easy to show that $\mathcal{F}(\ell_0) \ne \emptyset$, $\mathcal{F}(\ell_0)$ is a cone, and one of its boundaries is defined by $\{(x, y) : y = \frac{1}{\beta^{k^*}} x\}$. To identify its other boundary define

$$\ell_1 = \inf\{\ell(x, y) : (x, y) \in \mathcal{F}(\ell_0)\}$$

which is strictly less than ℓ_0, since $\forall (x, y) \in \mathcal{F}(\ell_0)$, $\ell(x, y) < k^* = \ell_0$.

Given $\ell_0, \ell_1, \ldots, \ell_{j-1}$, define $\mathcal{F}(\ell_{j-1})$ as the collection of states (x, y) satisfying the following two conditions:

$$\exists \Delta, \text{ such that } \beta(y-\Delta) = \frac{1}{\beta^{\ell_{j-1} - 1}}(x-\Delta) \tag{C.37}$$

$$3\Delta + \delta(3(y-\Delta)(\beta\delta)^{\ell_{j-1}}) \le (\beta\delta)^{\ell(x,y)} y \tag{C.38}$$

and define

$$\ell_j = \inf\{\ell(x,y) : (x,y) \in \mathcal{F}(\ell_{j-1})\}$$

Since $\ell_0 > \ell_1 > \ldots > \ell_j$ and $\ell_0, \ldots, \ell_j$ are positive integers, this process must stop in finite steps:

$$k^* = \ell_0 > \ell_1 > \ldots > \ell_j > \ldots > \underline{\ell}.$$

By construction, $\mathcal{F}(\ell_0), \ldots, \mathcal{F}(\underline{\ell})$ forms a partition of

$$\{(x,y) : x \leq y \leq \frac{1}{\beta^{k^*}} x\}.$$

Now, we are ready to spell out the strategy off the equilibrium path. As a first step, choose $(x,y) \in \mathcal{F}(\ell_0)$ and fix $p' < 3$. If $p' < 3 - 2\delta$, every high valuation consumer accepts the offer, expecting that the monopolist will charge 1 in the following round. If $3 - 2\delta \leq p' < 3$, then Δ portion of the high valuation consumers accepts the offer according to (C.35). Then, in the following round, the monopolist randomizes between $3 - 2\delta$ (the acceleration strategy) and 3 (the delay strategy) with probability θ assigned to 3 such that

$$3 - p' = \delta((1-\theta)(3-3) + \theta(3 - (3 - 2\delta)) = 2\theta\delta^2$$

to ensure that the high valuation consumers are indifferent between accepting and rejecting p'. By (C.36), the constructed strategies ensure that the monopolist has no incentive to deviate to p' from his equilibrium price of 3.

For the general case, choose $(x,y) \in \mathcal{F}(\ell_{j-1})$ and fix $p' < 3$. If $p' < 3 - 2\delta$, then every high valuation consumer accepts the offer, expecting that the monopolist will charge 1 in the following round. If $3 - 2\delta^{j'} \leq p' < 3 - 2\delta^{j'+1}$ for $j' < j$, then choose Δ' such that $\beta(y - \Delta') = \frac{1}{\beta^{j'-1}}(x - \Delta')$. The continuation game follows the equilibrium strategies constructed for $\mathcal{F}(\ell_{j'-1})$. If $3 - 2\delta^j \leq p' < 3$, then choose $\theta \in [0,1]$ so that

$$3 - p' = 2\delta^j \theta$$

to ensure that the high valuation consumer is indifferent between accepting and rejecting p'. Choose Δ according to (C.37), which represents the portion of high valuation consumers who accept p'. After Δ portion of consumers accept the offer, the continuation game is played according to the equilibrium strategy constructed for the states in $\mathcal{F}(\ell_{j-2})$. (C.38) ensures that the monopolist has no incentive to deviate to p' from his equilibrium offer of 3.

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